Coalgebras, Covarieties, Coequations

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Algebras and Coalgebras

Universal algebra's notion of *algebra* consists of

– a set X,
– a *signature* Σ of n-ary operations, and
– an *interpretation*

$$\sigma : X^{\text{arity}(\sigma)} \longrightarrow X$$

for each $\sigma \in \Sigma$.

In the language of categories, signatures are polynomial endofunctors

$$\coprod_{\sigma\in\Sigma} (-)^{\text{arity}(\sigma)} : \mathbb{S}\mathbf{ets} \longrightarrow \mathbb{S}\mathbf{ets},$$

and an interpretation is a function of the form

$$\coprod_{\sigma\in\Sigma} X^{\text{arity}(\sigma)} \longrightarrow X.$$

In general, given an endofunctor

$$\Sigma : \mathbb{D} \longrightarrow \mathbb{D},$$

a **Σ-algebra** is any arrow

$$\Sigma X \longrightarrow X.$$

Dually, given a functor

$$\Gamma : \mathbb{D} \longrightarrow \mathbb{D},$$

a **Γ-coalgebra** is an arrow

$$\alpha_X : X \longrightarrow \Gamma X.$$

A **Γ-coalgebra homomorphism** satisfies

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \alpha_X \downarrow & \# & \downarrow \alpha_Y \\ \Gamma X & \longrightarrow & \Gamma Y \end{array} .$$

The category of Γ-coalgebras and Γ-coalgebra homomorphisms is denoted $\mathbb{D}_\Gamma$, and the forgetful functor

$$U : \mathbb{D}_\Gamma \longrightarrow \mathbb{D}.$$

Typical Question: When is U monadic?

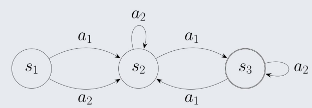
Examples From Mathematics

In general,
algebras model *datatypes*, and
coalgebras model *stateful systems*.

Automata. These are coalgebras for an endofunctor of the form

$$2 \times (-)^A : \mathbb{S}\mathbf{ets} \longrightarrow \mathbb{S}\mathbf{ets},$$

visualized as a set of *transitions between states* with labels.



$$\{s_1, s_2, s_3\} \longrightarrow 2 \times (\{s_1, s_2, s_3\})^{\{a_1,a_2\}}$$

Kripke Models. For a set of proposition letters **Prop**, these are functions

$$\alpha_W : W \longrightarrow \mathcal{P}(\mathbf{Prop}) \times \mathcal{P}(W).$$

$$\alpha_W(w)_1 = (\text{worlds accessible from } w)$$
$$\alpha_W(w)_2 = (\text{propositions } w \text{ believes}).$$

Graphs. Given $F : \mathbb{S}\mathbf{ets} \to \mathbb{S}\mathbf{ets}$, an **$F$-graph** is a coalgebra for the functor

$$\mathrm{Grph}(F) : \mathbb{S}\mathbf{ets}^2 \longrightarrow \mathbb{S}\mathbf{ets}^2,$$
$$(V, E) \longmapsto (1, FV).$$

For example, $\mathrm{Grph}((-)\times(-))$-coalgebras are equivalent to maps

$$\langle s, t\rangle : E \longrightarrow V \times V,$$

known as *quivers*. See [Rut00] for this example, and the recent [Jäk15].

Birkoff's Variety Theorem

Given a set $\mathcal{T}$ of equations in a signature Σ, let $\mathcal{T}^\top$ be the class of Σ-algebras that satisfy $\mathcal{T}$.

$\mathcal{T}^\top$ is a **variety**, meaning that it is closed under products, quotients, and subalgebras.

Theorem.(Birkoff's Variety Theorem) Let Σ be a signature. A class K of Σ-algebras is a variety if and only if it is of the form $\mathcal{T}^\top$ for some $\mathcal{T}$.

Question: Can we generalize to other algebras? What is the story for coalgebras?

Coequations and Covarieties

A right adjoint $H : \mathbb{D} \to \mathbb{D}_\Gamma$ to U takes D to its **cofree** Γ-coalgebra $HD = \langle S_D, \gamma_D\rangle$.

A **coequation** over $D \in \mathbb{D}$ is a regular subcoalgebra $\langle C, e_C\rangle \leq HD$. $\langle A, \alpha_A\rangle$ **satisfies** $\langle C, e_C\rangle$ if

$$\begin{array}{ccccc} A & & \langle A, \alpha_A\rangle & & \\ \forall f \downarrow & & \hat{f} \downarrow & \overset{\exists!}{\searrow} & \\ D & & HD & \underset{i}{\longleftarrow} & \langle C, e_c\rangle \end{array}$$

This can also be written $\langle A, \alpha_A\rangle \perp i$, read

"$\langle A, i\rangle$ is co-orthogonal to i."

A **covariety** is a full subcategory of $\mathbb{D}_\Gamma$ closed under

(a) coproducts,
(b) targets of epics, and
(c) sub-coalgebras.

This is the dual notion to *variety*.

The Covariety Theorem

Given a class $\mathcal{E}$ of coequations over D, let $\mathcal{E}_\perp$ be the full subcategory of Γ-coalgebras satisfying $\mathcal{E}$.

Categories of the form $\mathcal{E}_\perp$ are called **coequational**, and are examples of covarieties.

In the presence of sufficient structure, ie. in **co-Birkoff** categories, this is the whole story.

Theorem.(CoBirkoff Covariety Theorem) Suppose $\mathbb{D}$ is coBirkoff and $\Gamma : \mathbb{D} \to \mathbb{D}$ preserves regular monics. Then V is a covariety iff it is coequational.

See [AH00] for details. An earlier proof for $\mathbb{D} = \mathbb{S}\mathbf{ets}$ and bounded Γ is in [Rut00].

Some Applications

– Vast generalizations of modal logic [SBR16].
– Characterizing regular varieties of automata with varieties of languages [Sal+15].

[AH00] Steven Awodey and Jesse Hughes. "The Coalgebraic Dual Of Birkhoff's Variety Theorem". In: 2000.

[Rut00] J.J.M.M. Rutten. "Universal coalgebra: a theory of systems". In: *Theoretical Computer Science* 249.1 (2000).

[Jäk15] C. Jäkel. *A unified categorical approach to graphs*. 2015. arXiv: 1507.06328 [math.CO].

[Sal+15] J. Salamanca et al. "Regular Varieties of Automata and Coequations". In: *Mathematics of Program Construction*. Cham: Springer International Publishing, 2015, pp. 224–237.

[SBR16] J. Salamanca, M. Bonsangue, and J. Rot. "Duality of Equations and Coequations via Contravariant Adjunctions". In: *Coalgebraic Methods in Computer Science*. Springer International Publishing, 2016, pp. 73–93.